Calculus, Once Through Quickly

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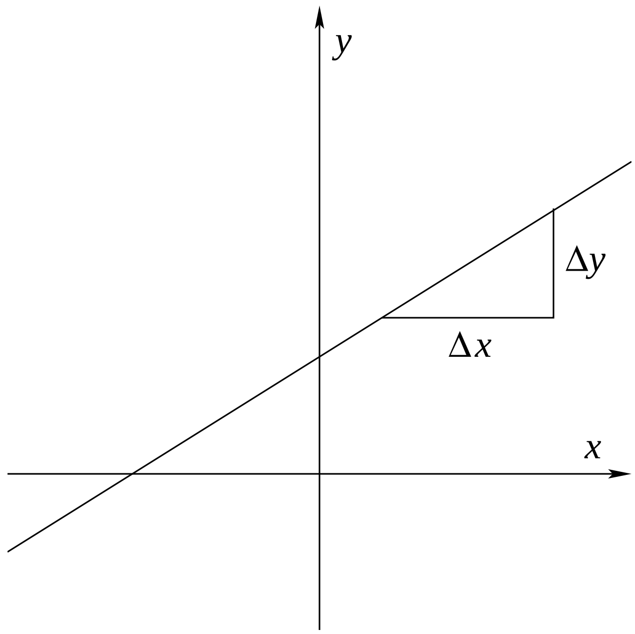
Calculus is too important to postpone until late high school or early college. Learning its nomenclature, techniques, and especially its concepts is a prerequisite to fully understanding motion of any kind: falling objects, changes in population, flows of money in the economy, *etc.* This was, after all, why Newton and Leibniz independently and nearly simultaneously invented it: to write down and solve equations of motion.

This mini-textbook attempts to lay out the foundations of differential and integral calculus in a form that will be usable by a bright and interested 12 to 16-year-old. The only prerequisites are a familiarity working with ordinary algebra (e.g., solving $2x + 3 = 7$ and $\sqrt{y^2 + 4^2} = 5$), and two-dimensional plotting in a Cartesian $x - y$ plane. Subjects not covered here include trigonometry, complex numbers, differential equations, and numerical methods.

This text uses examples to introduce and explain concepts, then backs them up with general results that are derived with a moderate degree of rigor. To get full benefit from this material, and to have any hope of committing it to long-term memory, a student has to actively follow along with pencil and paper. Math is not a spectator sport!

Slope

The concept of slope is familiar to anyone who traverses our planet. Mathematicians have a quantitative definition that is consistent with most other uses: verbally, "rise over run." Elaborating: this is the ratio of vertical change to horizontal change. When lines are drawn on a conventional $x - y$ Cartesian coordinate system, it looks like this:

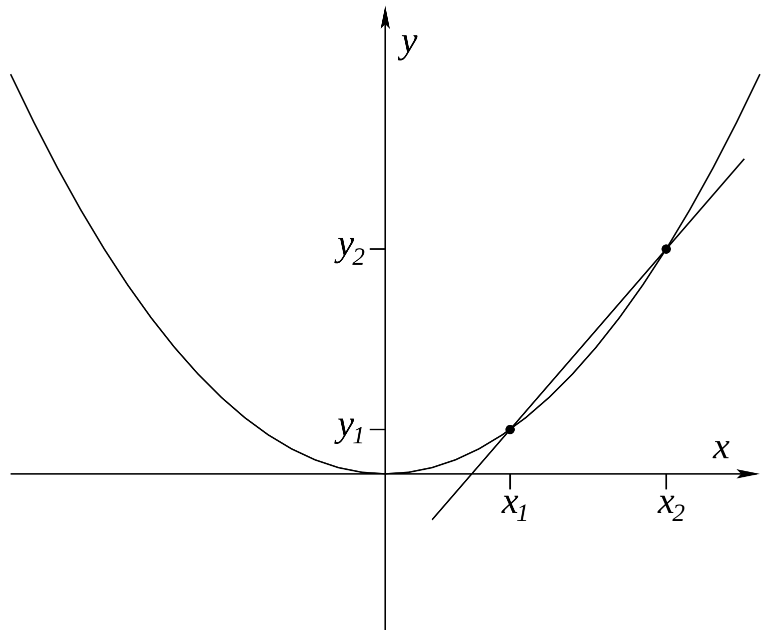


Here Δ is the Greek equivalent of the capital letter D, is pronounced "delta," and represents *difference in*. The slope of a line is $\Delta y/\Delta x$, the rise (Δy) divided by the run (Δx).

The line shown has positive slope, normally expressed with positive Δy and positive Δx. If you changed your mind and wanted to represent Δx as negative (considering the end points from right to left), Δy would also come out negative, and their ratio–the slope–is still positive. To get a negative slope, the line has to trend down (decreasing y) as x increases (moves to the right).

Slope of a Parabola

Now consider a very simple *curved* line, $y = x^2$ shown here. Start at the point (x_1, y_1), where of course $y_1 = x_1^2$ (read x subscript 1, squared). Move a distance h to the right, to the point (x_2, y_2), where $x_2 = x_1 + h$, and $y_2 = (x_1 + h)^2$.



As long as h is not zero, the slope of the line that connects these two points is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{(x_1+h)^2 - x_1^2}{h} \\ &= \frac{(x_1^2 + 2x_1 h + h^2) - x_1^2}{h} \\ &= \frac{2x_1 h + h^2}{h} \\ &= 2x_1 + h\end{aligned}$$

It looks clear that as h gets smaller and smaller, the slope of the connecting line approaches a *limiting value*, and that value is $2x_1$. It's also clear that just setting $h = 0$ in the first equation is useless, since that leads to an undefined $0/0$ formula. The end result of the limit is correct: the slope of the parabola $y = x^2$, evaluated at the point x_1, is indeed $2x_1$.

Functions

The relationship $y = x^2$ from the previous section is an example of a *function*. Functions have an *argument*, the number they take in, and a *result*, that they spit out. We can give a function a letter name, such as f. Then we can define this function like this: $f(x) = x^2$. Read this as "f evaluated at x is x squared." The set of input arguments for which they provide a valid answer is its *domain*. By their very nature, a function yields a unique result for every element of its domain. In this example, 2 is a valid argument for $f(x)$, and the result $f(2) = 2^2 = 4$.

We can construct new functions in terms of existing functions. Suppose we are given one function f. A new function g can be expressed in terms of f, even without specifying what f is. For example, we could define $g(x) = 1 - f(x)$. We could also write this in a shorthand form $g = 1 - f$, as long as we remember that f and g are functions, not numbers. As long as we eventually learn what f is, g is just as valid a function as f, and in fact it has the same domain as f.

Derivative

The slope of a function is its derivative. The notation and precise definition is as follows: Suppose f is a function of x, so that for real values of x (at least within an interval of interest) there is a single real result $y = f(x)$. Define a new function f' such that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If the limit is well defined, this function f' is called the derivative of f.

The best formal definition of the limit is based on a challenge/response test. Suppose I tell you that

$$\lim_{h \to 0} g(h) = b$$

You can pick any (arbitrarily small) positive number ε, and challenge me to find an $h_1 \neq 0$ such that $|g(h_1) - b| < \varepsilon$. If the limit is correct, I can always do so.

Take as an example the limit from the parabola slope example.

$$g(h) = 2x + h$$

I claim the limit as $h \to 0$ is $2x$. For any positive but arbitrarily small ε you choose, I can choose $h = \varepsilon/2$. The difference $|g(x_1) - b|$ becomes $\varepsilon/2$, which is less than what you challenged me to meet. My original statement that the limit is $2x$ passes the challenge/response test.

Connection to Physics

Any object undergoing constant acceleration, including one in free fall at the surface of the earth, traces out a parabola.

$$y(t) = -\frac{1}{2}gt^2$$

Where y is the vertical position of the object. Taking the derivative of this equation gives us

$$y'(t) = -gt$$

And we call y', the derivative of position, v for velocity. Taking the derivative again, we get

$$y''(t) = -g$$

The double prime is pronounced "second derivative," and clearly means the derivative of the derivative. The physics name for this second derivative of position, or first derivative of velocity, is (did you guess it?) acceleration.

A bevy of first-year physics equations and problem sets start from this simple starting point. Sometimes they are presented without their basis in calculus, which can make them seem more mysterious and less interrelated than they really are.

Notation for Derivative

Given the long history of the derivative, and its utility across a broad range of applied mathematics and science, it is not surprising that a variety of notations for the derivative have been developed. Here are a few.

$$g = f'$$

$$\frac{df}{dx} = g \qquad \text{(Leibniz notation)}$$

$$D(f) = g$$

There are similar notations for second derivative:

$$h = f''$$

$$\frac{d^2 f}{df^2} = h$$

$$D^2(f) = h$$

Derivative of $1/x$

If $f(x) = 1/x$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} \\ &= -\frac{1}{x^2} \end{aligned}$$

Derivative of x^n

The derivative of x is 1, and the derivative of x^2 was computed as our original example: it is $2x$. A similar process will derive that the derivative of x^3 is $3x^2$. The pattern leads us to (correctly) hypothesize that for any positive integer n, the derivative of x^n is $n \cdot x^{n-1}$. The usual proof is based on induction: assume it's true for $n = m$, and show that it's true for $n = m + 1$.

Properties of Derivative

Suppose a is an arbitrary constant, and f is a function. Define $u(x) = af(x)$. Try now to compute the derivative of u, also known as u'.

Answer: $u'(x) = af'(x)$

Suppose f and g are arbitrary functions, and define $u(x) = f(x)+g(x)$. Try now to compute the derivative of u, also known as u'.

Answer: $u'(x) = f'(x) + g'(x)$

Combined, these properties show that the derivative is a *linear operator*. These relationships can also be written $D(af) = aD(f)$ and $D(f+g) = D(f) + D(g)$.

Derivative of a Product

Let's compute the derivative of the product of an arbitrary pair of functions. This gets a bit messy, but two things are important: the result, and the fact that the result is *derived*, not assumed.

Assume we know already that f' is the derivative of f, and g' is the derivative of g. Define $u(x) = f(x)g(x)$, and seek u'. The formula is

$$u'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Add and subtract $f(x+h)g(x)$ in the above numerator, in order to collect terms that we know from the expressions for f' and g'.

$$\begin{aligned} u'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h} \\ &= f(x)g'(x) + f'(x)g(x) \end{aligned}$$

Derivative of $\sqrt{x}$

Consider the case $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x}$. Their product is $u(x) = x$, for which $u'(x) = 1$. Apply the rule for the derivative of a product to get

$$f'(x) \cdot \sqrt{x} + \sqrt{x} \cdot g'(x) = 1 \quad .$$

Since $f'(x)$ and $g'(x)$ are the same, rearrange this equation to get

$$f'(x) = \frac{1}{2\sqrt{x}} \quad .$$

This continues the pattern $D(x^a) = a \cdot x^{a-1}$. It is difficult to prove rigorously, but that equation is correct for all constant values of a.

Chain Rule

Start with two functions, $f(x)$ and $g(x)$, and chain them together to form $u(x) = f(g(x))$. f is called the *exterior function*, and g is called the *interior function*. The chain rule expresses the derivative, $u'(x)$, in terms of f, g, and their derivatives. It can be derived like the product rule above. The result is

$$u'(x) = f'(g(x)) \cdot g'(x)$$

Expressed in words: the derivative of the exterior function applied to the interior function, multiplied by the derivative of the interior function.

An example will make this more clear. Define $f(x) = 1/x$, and $g(x) = x^2 + 1$, so the chained function

$$u(x) = f(g(x)) = \frac{1}{x^2+1} \quad .$$

What is $u'(x)$? The techniques and examples so far can give us $f'(x) = -1/x^2$, and $g'(x) = 2x$. The chain rule combines these results:

$$\begin{aligned} u'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{-1}{(x^2+1)^2} \cdot 2x \\ &= \frac{-2x}{(x^2+1)^2} \end{aligned}$$

Try another example yourself: $f(x) = x^2$, and $g(x) = x + 2$. After you run these functions through the chain rule, you can check yourself by expanding $(x+2)^2 = x^2+4x+4$, and taking the derivative term by term (valid because we already showed that the derivative is a linear operator).

Application

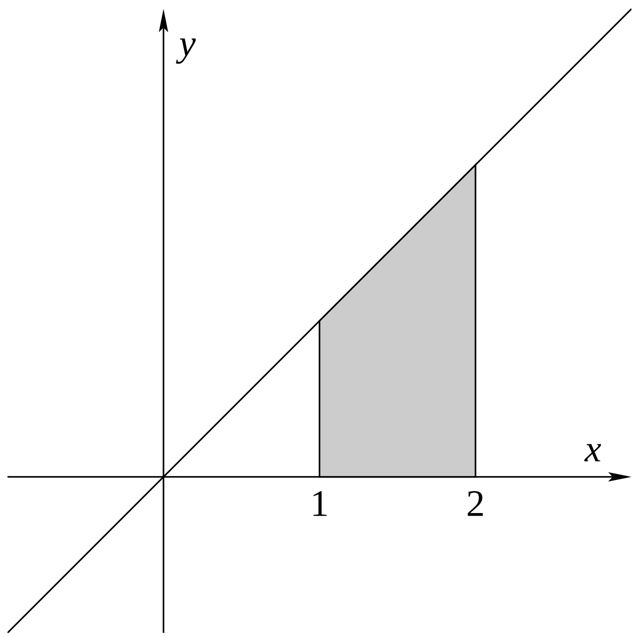
When faced with a new function to differentiate, it is in principle possible to start fresh with the definition of a derivative, in terms of limits. Practically, it is usually quicker and less error-prone to compute the derivative based on the relations we just derived, just as was done for $1/(x^2 + 1)$ in the previous section. Practice this process with these functions:

$$\frac{1}{x^2} \qquad \frac{1}{\sqrt{x^2+1}}$$

$$x\sqrt{x^2+1} \qquad \sqrt{x^2+1}$$

Area of a Triangle

We turn now to the second major branch of calculus, which is concerned with finding the area under curves. Let's get started with a specific example, where everyone should be able to compute areas using ordinary geometry. Take the function $y = x$, and select a couple of points to define an interval along the x axis: from $x_1 = 1$ to $x_2 = 2$. What is the area of the shaded region?



The traditional geometry formula is $\frac{1}{2}b(h_1 + h_2)$, where in this case the base is 1, and the two heights are 1 and 2. The area is therefore 3/2.

Now generalize just a little bit: what is the area in terms of x_1 and x_2? The trapezoid base is $x_2 - x_1$, $h_1 = x_1$, and $h_2 = x_2$. The final area is

$$\tfrac{1}{2}(x_2 - x_1)(x_1 + x_2) = \tfrac{1}{2}x_2^2 - \tfrac{1}{2}x_1^2$$

That second form is interesting: you can interpret it as the area of the triangle from 0 to x_2, minus the area of the triangle from 0 to x_1.

Notation for Definite Integral

Let's introduce some notation: what we have just computed is called a *definite integral.*

$$\int_{x_1}^{x_2} x dx = \tfrac{1}{2}x_2^2 - \tfrac{1}{2}x_1^2$$

where the left hand side is read "integral from x_1 to x_2 of x, dx."

Indefinite Integral

That previous formula shows a very general pattern, that is more clear if we name two of the functions involved. Define $f(x) = x$, and define $g(x) = \frac{1}{2}x^2$. Then

$$\int_{x_1}^{x_2} f(x)dx = g(x_2) - g(x_1)$$

When f and g have a relationship like this, we say that g is an *indefinite integral* of f, and specify it like this:

$$\int f(x)dx = g(x)$$

The left hand side is read "integral of $f(x)$ dx," "integral of $f(x)$ with respect to x," or sometimes just "integral of $f(x)$."

For every integrable function $f(x)$, there is a whole family of indefinite integrals $g(x)$. If you know one such function g, pick any constant C, and construct $u(x) = g(x) + C$. Then

$$\begin{aligned}\int_{x_1}^{x_2} f(x)dx &= u(x_2) - u(x_1) \\ &= (g(x_2) + C) - (g(x_1) + C) \\ &= g(x_2) - g(x_1)\end{aligned}$$

so u is also an indefinite integral of f. That constant C is called a *constant of integration.* In the particular example above, people sometimes write

$$\int x dx = \tfrac{1}{2}x^2 + C$$

Derivative and Integral

Take the general expression for definite integral, and find out what happens if we integrate from a to $a+h$, and take a limit as h goes to 0.

$$\int_a^{a+h} f(x)dx = g(a+h) - g(a)$$

It's pretty clear what's supposed to happen to the left hand side: the area integrated gets narrower and narrower, until the only thing that matters is the value of f at a itself. The area of that almost-rectangle is $h \cdot f(a)$. Divide the expression through by h, to get

$$f(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

Where have we seen that before? (Hint: turn back a few pages.) Oh, yes. This is the definition of derivative, and means $g' = f$. This interrelationship between derivative and integral is known as the Fundamental Theorem of Calculus.

Review of Integration

Integrating and differentiating are opposites. To find the indefinite integral of f, find a function g whose derivative is f, then

$$\int f dx = g$$

There are an infinite set of such functions g, that differ from each other by a constant. Any of these functions will do. Once you have that function g in hand, you actually find the area under a section of the curve f from $x = x_1$ to $x = x_2$ like this:

$$\int_{x_1}^{x_2} f(x)dx = g(x_2) - g(x_1)$$

Applying this to the original problem, suppose we are given $f(x) = x$. What function's derivative is x? Answer: $g(x) = \frac{1}{2}x^2 + C$, for any constant

C, and it's easiest to take $C = 0$. Now the area under the function from 1 to 2 is evaluated as $g(2) - g(1) = \frac{1}{2}(2)^2 + \frac{1}{2}(1)^2 = \frac{3}{2}$.

The process of finding the integral of a new function is rarely as prescriptive as that of finding a derivative. Derivatives can normally be found by construction. Solving integrals often takes perseverance, insight, luck—or looking it up in a book.