Resonant $n > 0$ Modes Trapped by a Dielectric in a Coaxial Line

Lawrence R. Doolittle
Tunnel Dust, Inc.
Walnut Creek, California, USA

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1 Introduction

The simplest possible coaxial window geometry has a dielectric material inserted between coaxial metal cylinders, which extend undisturbed beyond the ends of the window, as shown in figure 1. The dielectric reduces the impedance of the line, so some form of impedance match is required.

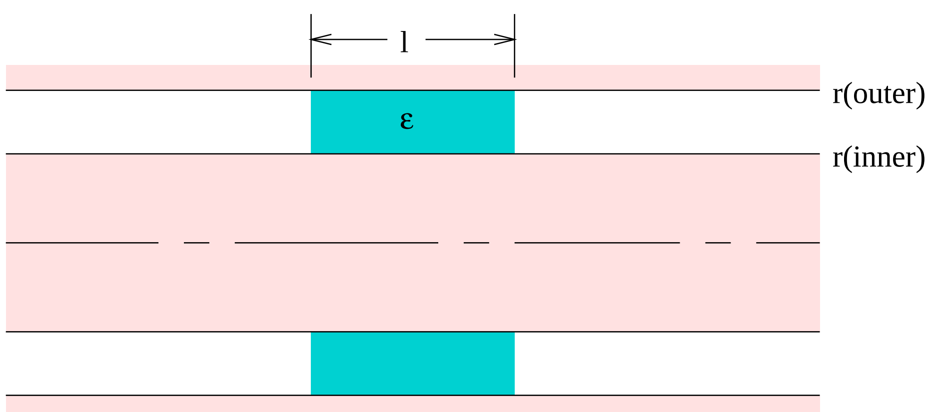

Figure 1: Coaxial window geometry

One approach is to use a two-step quarter-wave match, from a high ($\sim 50\,\Omega$) impedance line to low ($\sim 5\,\Omega$) impedance window. This is simple, compact, and leaves the region in the vicinity of the window undisturbed from the form shown in figure 1.

Scaling laws provide an incentive to make a high power window as large as possible, since (at constant l, at least) losses scale inversely with radius, and both heat flux and temperature rise scale inversely with the radius squared. One phenomenon that limits how large a window can be made is the presence of higher order modes in the structure. Conventional wisdom is to limit the average circumference to one wavelength, so that these modes cannot propagate.

This rule makes sense, and the derivation below will confirm it for the vacuum section of the line. Applying that rule to the dielectric loaded section of the line is too conservative. While the dielectric can trap modes below the cutoff frequency of the vacuum line, those narrow band resonances can be computed (analytically) and placed away from frequencies of operation.

2 HOM Math

As discussed in Langmuir[1], modes of a coaxial line have a radial variation given by

$$R(r) = A_n J_n(kr) + B_n N(kr) \quad .$$

The TM boundary conditions at $r = r_i$ and $r = r_o$ give

$$\frac{N_n(k_r r_i)}{J_n(k_r r_i)} = \frac{N_n(k_r r_o)}{J_n(k_r r_o)} = -\frac{A_n}{B_n}$$

and the corresponding conditions for TE waves are

$$\frac{N_n'(k_r r_i)}{J_n'(k_r r_i)} = \frac{N_n'(k_r r_o)}{J_n'(k_r r_o)} = -\frac{A_n}{B_n} \quad .$$

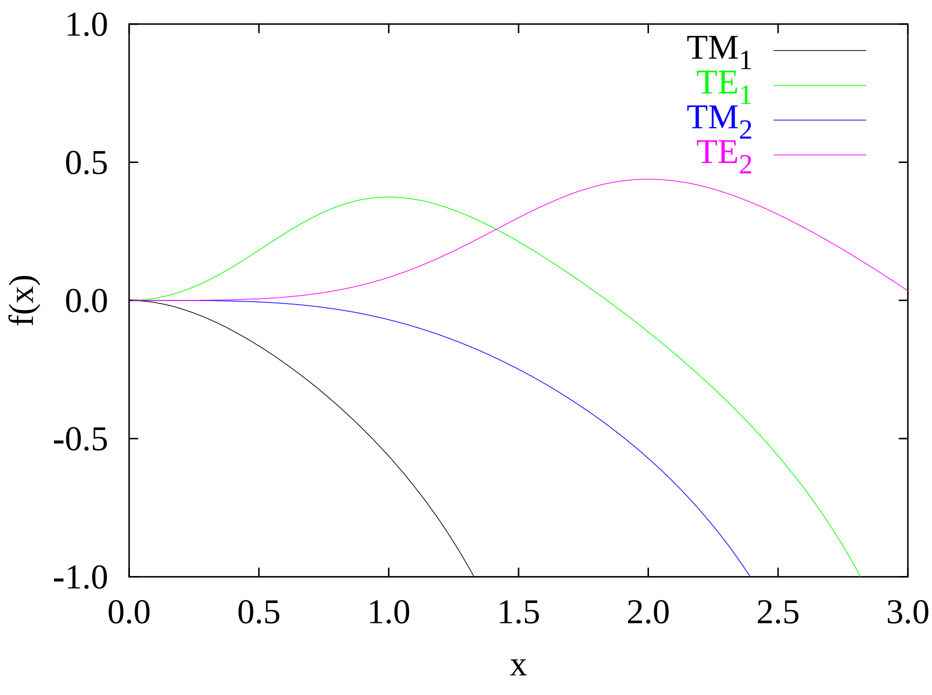

Figure 2: Bessel function ratios

When r_i is within a factor of 2 of r_o, the family of modes with the lowest cutoff frequencies are TE modes. These have $k_r = n/r_a$, where n is the azimuthal quantum number, and $r_a \approx (r_i + r_o)/2$, but r_a really has a weak dependence on both n and r_o/r_i.

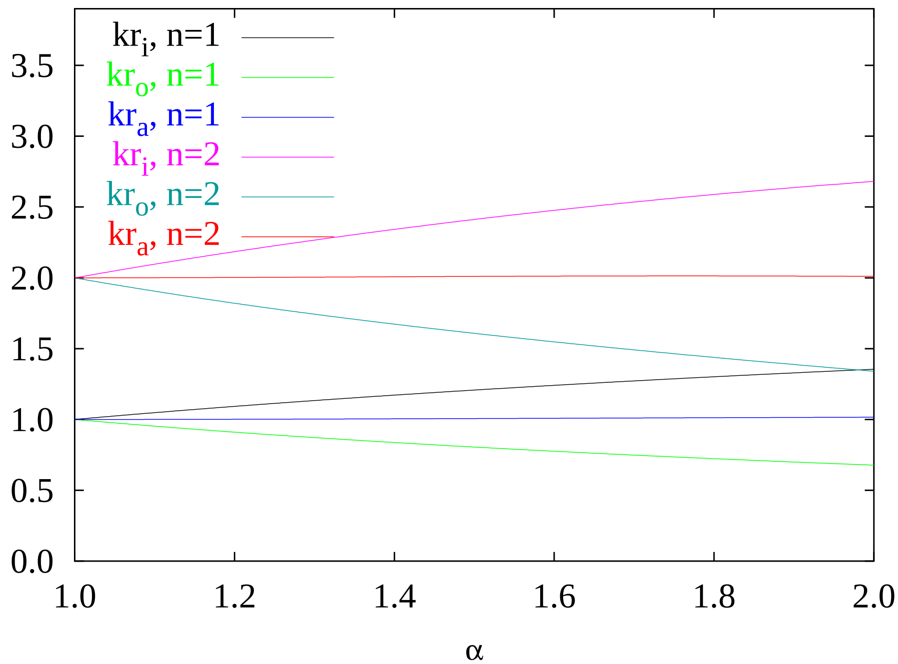

Figure 3: Normalized transverse separation constants

The fact that small values of k can apply to even closely spaced conductors ($r_i \approx r_o$) is possibly surprising. Inspection of a graph of the above Bessel function ratios (figure 2, in which the TE$_n$ curves show $N_n'(x)/J_n'(x)$, and the TM$_n$ curves show $N_n(x)/J_n(x)$) makes it clear how this can work. The peaks in the TE style curves represent cases where two closely spaced x values can have the same $f(x)$. This arrangement does not happen with the TM curves. Representing such solutions ($f(kr_i) = f(kr_o)$) of these graphs in terms of $\alpha = r_o/r_i$ gives us the curves in figure 3.

Treat a section of dielectric of length l in a coaxial line with inner radius r_i and outer radius r_o, as depicted in figure 1. Consider behavior at angular frequency ω, for which the free-space propagation constant is $k_0 = \omega/c$, and the corresponding unguided propagation constant in a medium of dielectric constant is $k_\epsilon = \sqrt{\epsilon}k_0$. Assume $k_0 < k_r < k_\epsilon$, so HOMs don't propagate in the $\epsilon = 1$ section, but do propagate inside the dielectric. Then the wave equation's solution has z dependence $\sin k_1 z$ and/or $\cos k_1 z$ in the dielectric, and $e^{\pm az}$ in the adjoining vacuum. The separation constants satisfy $k_1^2 = k_\epsilon^2 - k_r^2$ and $a^2 = k_r^2 - k_0^2$.

If we match H_z and dH_z/dz on the dielectric-vacuum boundaries $z = \pm l/2$, and take the $\cos z$ style solution in the dielectric, then we get relations

$$\cos k_1 \frac{l}{2} = A$$

$$-k_1 \sin k_1 \frac{l}{2} = -aA.$$

Eliminating A from these equations, we get the condition for resonance

$$l = \frac{2}{k_1} \tan^{-1} \frac{a}{k_1}.$$

Note that the multi-valued nature of $\tan^{-1}$ gives a whole class of solutions, separated by π in the arctangent. Changing from cos (even) solutions to sin (odd) solutions produces solutions that are spaced $\pi/2$ from the first set. Combine this with the relations

$$k_1 = \frac{1}{r_a}\sqrt{\epsilon t^2 - n^2}$$

$$a = \frac{1}{r_a}\sqrt{n^2 - t^2}$$

where $t = \omega r_a/c$. A suitably normalized form of the result is

$$\frac{l}{r_a} = \frac{2}{\sqrt{\epsilon t^2 - n^2}}\left(\tan^{-1}\sqrt{\frac{n^2 - t^2}{\epsilon t^2 - n^2}} + m\frac{\pi}{2}\right) \quad ,$$

where m is the longitudinal quantum number. This relation is displayed in figure 4. It is clear that trapped (resonant) modes can only exist if the normalized frequency t satisfies $n/\sqrt{\epsilon} < t < n$.

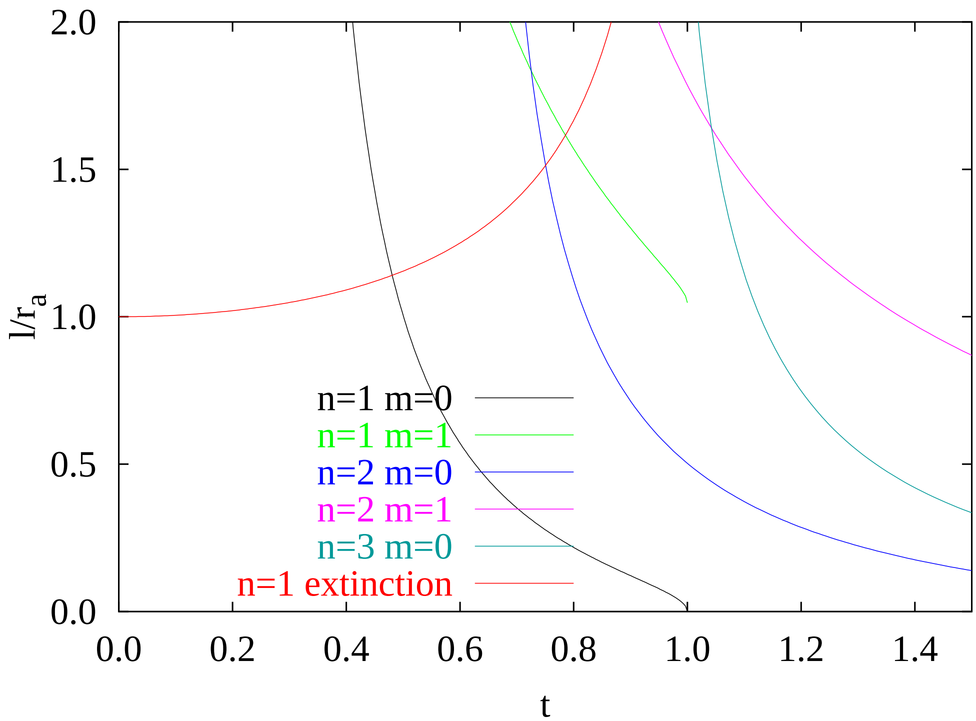

Figure 4: Normalized resonant frequencies for $\epsilon = 10$

3 Example

The preceding math may seem a little abstract and opaque, so here is an example that may clarify the implications.

If we work with Al_2O_3 ceramic that has $\epsilon = 10$, and the window assembly fits in a traditional $50\,\Omega$ line, a magic value of $\alpha = r_o/r_i$ for the window region is 1.30, because the $60 \ln \alpha = 15.8\,\Omega$ vacuum line impedance can be used as a quarter-wave matching section to the $60/\sqrt{\epsilon} \cdot \ln \alpha = 5\,\Omega$ dielectric loaded section.

Suppose a window is needed for 700 MHz. To get a reasonable extinction length for HOMs, we should choose r_a somewhat less than the critical value of $c/2\pi f = 6.82$ cm. Capriciously choosing a value of 20 cm for the extinction length, we derive a value of 6.45 cm for r_a. Thus the normalized frequency t in the section above is 0.947.

Solving for r_i and r_o in terms of $r_a = (r_i + r_o)/2$ and $\alpha = r_o/r_i$, we get $r_i = 2r_a/(1+\alpha) =$ 5.61 cm and $r_o = 7.30$ cm.

Inserting all these parameters to find l for various modes gives as set of lengths to avoid for the window. For $n = 1$ modes, the first two lengths to avoid are 0.52 and 7.71 cm. The first $n = 2$ mode will line up with the operation frequency if the window length is 3.88 cm.

Suppose the window is made 3.0 cm long, which normalizes to $l/r_a = 0.456$. Although the equation for l/r_a is not analytically invertible, it is easy to solve for the resonant frequency numerically, and get $t = 0.641$, or 474 MHz. The field pattern for this mode, and and the next two lowest frequency modes, are shown in figure 5. With a loop probe some distance away from the window, these modes could presumably be excited on purpose. The Q_0 of such trapped modes would be a useful low power indicator of window losses.

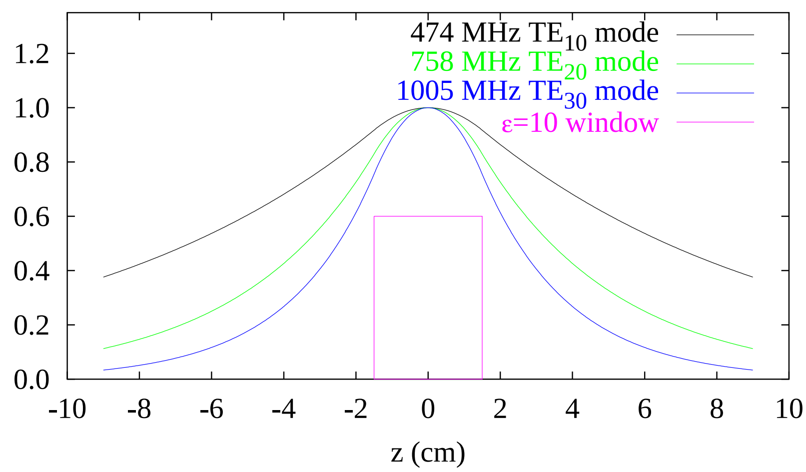

Figure 5: Longitudinal H_z dependence of resonant modes discussed in the example

References

[1] Robert V. Langmuir, **Electromagnetic Fields and Waves**, McGraw-Hill, New York, 1961, section 12.5.